Iterated forcing, Part 2: CS products and halving

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Outline

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Notation

Recall:

- $P_3 = P_2 * Q_2 = Q_0 * Q_1 * Q_2$.
- $G_2 \subseteq P_2$ generic over V, $G(2) \subseteq Q_2$ generic over $V[G_2]$.
  $G_2 * G(2) \subseteq P_2 * Q_2 = P_3$ generic over V.

For example: We want to find a model where
$2^{\aleph_0} = \kappa = \mathrm{non}(\mathcal{M})$, i.e., every "small" set is meager, and the smallest nonmeager set is of size κ.
So we construct an iteration $(P_\alpha, Q_\alpha : \alpha < \kappa)$ with last element P_κ, where in each stage α the forcing notion Q_α will ...

- ... add a new real η_α
- ... add a new meager set M_α covering all reals in $V[G_\alpha]$.

In the end, we will have (at least) κ many reals, and every set of size $< \kappa$ will have appeared in an intermediate universe $V[G_\alpha]$ (not obvious, work a little bit), so it will be covered by the meager set M_α in the next universe $V[G_{\alpha+1}]$.

Why iterations? - continued

More generally:
We want to force a statement of the form $\forall X\ \exists Y : \varphi(X, Y)$, where

- X is usually a set with few elements (e.g., a small set of reals, or a small family of measure zero sets),
- and Y will be an object demonstrating that X is small in some other sense (e.g., a meager set covering X, or a new real not contained in any element of X)

We start by using a forcing Q_0, which adds an object Y_0 taking care of all $X \in V$.
But then we get new objects X, so we have to force again with Q_1, to get a Y_1 taking care of those X.
etc.

At the end, after κ many steps, we (hopefully) catch our tail and have taken care of all X.

Why not iterations?

- Finite support: can only handle ccc forcing notions.
- Finite support: always adds Cohen reals. (However, see tomorrow's lecture)
- Countable support: CH after $\alpha + \omega_1$ steps. Cannot get $2^{\aleph_0} > \aleph_2$.
- other supports, other limits: (not in this lecture)

Outline

Definition

Let $(Q_i : i \in I)$ be a family of forcing notions. The countable support product $Q = \prod_{i \in I} Q_i$ is the set of all partial functions p with finite or countable domain $\subseteq I$ satisfying $p(i) \in Q_i$ for all i.

Q is naturally ordered by the pointwise order. Each factor Q_i is naturally embedded into Q.

If $G \subseteq Q$ is generic, then its projection $G(i) \subseteq Q_i$ is generic for Q_i over V.

The products considered in this talk will always have $\aleph_2$-cc. (All Q_i will be of size $2^{\aleph_0}$. Now use CH and a Δ-system argument.)

Why not CS products?

Problems

- $G(i)$ is not generic over $V[G(j)]$.
  (Actually: $G(i)$ *is* generic over $V[G(j)]$, but only for the forcing $Q_i \in V$. Often we have a definition of Q_i, and we can evaluate this definition in $V[G(j)]$ yielding a name $\underset{\sim}{Q}'_i$; then $G(i)$ is usually not generic for $\underset{\sim}{Q}'_i[G(j)]$ over $V[G(j)]$.
- Not clear if the product will preserve $\aleph_1$.

Examples

- The CS product of infinitely many Cohen reals collapses ω_1.
- The CS product of infinitely many unbounded reals collapses ω_1.
- The product of 2 (!) proper forcing notions may collapse ω_1. (ZFC example)

Outline

PLAN On the following slides I will motivate the technique of “creatures” with “halving”, which was one ingredient in a recent paper of A.Fischer-G-Kellner-Shelah.
(not a new technique)

DISCLAIMER To make things more transparent, I will lie occasionally, by downplaying or ignoring important details.

WARNING Still, a lot of technical background needs to be digested.

Motivation

Fix a sequence $\bar{J} = (J_n : n \in \omega)$ of intervals of natural numbers, which are far apart and grow quickly:

$$\cdots \ll \min J_n \ll \max J_n \ll \min J_{n+1} \ll \cdots$$

We want to add a generic function g where $g(n) \subseteq 2^{J_n}$ is a set of large relative measure (say, more than $(1 - {}^{1}/_{2^n})$).
The set $\{x \in 2^\omega \mid \forall n : x \restriction J_n \in g(n)\}$ has positive measure, so $E_g := \{x \in 2^\omega \mid \forall^\infty n : x \restriction J_n \in g(n)\}$ has measure 1.
We want this set to avoid all ground model reals; “iterating” our forcing many times this will tend to make non(null) big.
(non(null) = the smallest size of a non-Lebesgue-null set)

We let $\mathrm{LARGE}_n := \{A \subseteq 2^{J_n} : {}^{|A|}/_{|2^{J_n}|} > 1 - {}^{1}/_{2^n}\}$.

A generic null set, part 2

We want to add a generic function g with $g(n) \subseteq 2^{J_n}$ a set in $\mathrm{LARGE}_n := \{A \subseteq 2^{J_n} : {|A|}/{|2^{J_n}|} > 1 - 1/2^n\}$.

Definition

Let $Q^{\bar{J}}$ be the set of all $p = (k^p, s^p, \bar{C}^p)$, where

1. $s^p = (s^p_0, \ldots, s^p_{k^p-1})$, $\forall i < k^p : s_i \in \mathrm{LARGE}_i$.
2. $\bar{C} = (C_n : n \geq k)$; $\forall n : C_n \subseteq \mathrm{LARGE}_n$.
3. $\limsup_{n\to\infty} \|C_n\|_n = \infty$, where $\|C\|_n = {\log\text{(some reasonable measure of } C)}/{\min J_n!!}$.

(Here $x \mapsto x!!$ is some sufficiently fast growing function.)

The sets C_n are called "creatures", their elements "possibilities". (Namely: possibilities for fragments of the generic.)
Any generic filter G defines a generic function g, and the set $E_g := \{x \in 2^\omega \mid \forall^\infty n : x{\restriction}J_n \in g(n)\}$ has measure 1.
For every old real $x \in 2^\omega$, the set of all conditions p satisfying "there are infinitely many n such that $x{\restriction}J_n$ avoids all $A \in C^p_n$" is dense (explain why!); hence $x \in 2^\omega \setminus E_g$, a null set.

Lemma

The forcing $Q^{\bar{J}}$ has "continuous reading of names", even "rapid reading". (=Lipschitz reading)
More explicitly: For any name $\underset{\sim}{x} \in 2^\omega$, and any condition p there is a stronger condition q such that:

- *For all n, the value of $\underset{\sim}{x} \restriction \max(I_n)$ will depend only on $g \restriction \max(I_n)$.*

Moreover, if we demand the above only for $n \geq n_0$, then we may also demand that p and q agree on all creatures below n_0.

Proof.

A fusion argument. (blackboard?)

Corollary

Let $\bar{J}$ and $\bar{J}'$ be "very disjoint" sequences of intervals, and let $G \times G'$ be generic for the forcing $Q^{\bar{J}} \times Q^{\bar{J}'}$. Then the set $2^\omega \setminus E_g$ will cover not only all reals from V, but also all reals from $V[G']$.
every $Q^{\bar{J}}$-name $\underset{\sim}{x} \in 2^\omega$

(For the proof, we have to work a bit with the norms.)

By modifying the forcing notion $Q^{\bar{J}}$ a little bit, we get the following stronger version:

Theorem

Assume GCH for simplicity, κ uncountable and regular.
Let $P = \prod_{i<\kappa} Q_i$ be a countable support product of forcing notions Q_i, each isomorphic to (the same) $Q^{\bar{J}}$.
Then each coordinate i^* comes conceptually "after" all the other coordinates. That means:

> *Whenever $\underset{\sim}{x}$ is a $\prod_{i \neq i^*} Q_i$-name of a function in 2^ω, then x avoids the measure 1 set E_{g^*} (where g^* is the generic function added by Q_{i^*}).*

As a consequence, $\Vdash_Q$ non(null) $\geq \kappa$.

WARNING Everything so far was just a warm-up.
The serious stuff starts **now**.

We start by recalling the description of the generic null set, and change it to a generic meager set.

What we did 10 minutes ago: generic null

Motivation

Fix a sequence $\bar{J} = (J_n : n \in \omega)$ of intervals of natural numbers, which are far apart and grow quickly:

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We want to add a generic function g where $g(n) \subseteq 2^{J_n}$ is a set of large relative measure (say, more than $(1 - 1/2^n)$).
The set $\{x \in 2^\omega \mid \forall n : x{\restriction}J_n \in g(n)\}$ has positive measure, so $E_g := \{x \in 2^\omega \mid \forall^\infty n : x{\restriction}J_n \in g(n)\}$ has measure 1.
We want this set to avoid all ground model reals; “iterating” our forcing many times this will tend to make non(null) big.
(non(null) = the smallest size of a non-Lebesgue-null set)

A generic meager set

Motivation

Fix a sequence $\bar{I} = (I_n : n \in \omega)$ of intervals of natural numbers, which are far apart and grow quickly:

$$\cdots \ll \min I_n \ll \max I_n \ll \min I_{n+1} \ll \cdots$$

We want to add a generic function g, defined on $\bigcup_n I_n$.
The set $R_g = \{x \in 2^\omega \mid \exists^\infty n : x{\restriction}I_n = g{\restriction}I_n\}$ is residual (co-meager), its complement
$M_g := \{x \in 2^\omega \mid \forall^\infty n : x{\restriction}I_n \neq g{\restriction}I_n\}$ is meager.
We want the set M_g to contain all ground model reals.
This means that in our forcing conditions we must have the possibility to remove $x{\restriction}I_n$ from almost all C_n.
This will make fusion more difficult.

A generic meager set, part 2

We want to add a generic function g defined on $\bigcup_n I_n$,

Definition

Let $Q^{\bar{I}}$ be the set of all $p = (k^p, s^p, \bar{C}^p, \bar{d}^p)$, where

1. $s^p = (s^p_0, \ldots, s^p_{k^p-1})$, $\forall i < k^p : s_i \in 2^{I_i}$.
2. $\bar{C} = (C_n : n \geq k)$; $\forall n : \emptyset \neq C_n \subseteq 2^{I_n}$.
3. $d^p = (d_n : n \geq k)$, each $d_n \in \mathbb{R}^+$.
4. $\liminf_{n\to\infty} \|C_n\|_n = \infty$, where $\|C\|_n = {}^{\log(|C| - d_n)}/_{\min J_n!!}$.

$q \leq p$ means all the obvious things: k becomes bigger, s becomes longer (inside the appropriate C_i), the C_i shrink, and $d^q_n \geq d^p_n$ for all $n \geq k^q$.

Halving and unhalving

Halving = Take 50% of all our possessions (not counting those which are already hidden), and hide them in a secret stash. Logarithmically speaking, we have lost almost no money. (At most one zero, from 1000 million to 500 million)
Concretely: Halving a creature (C_n, d_n) means: replace d_n by $d'_n := d_n + \frac{1}{2}(|C_n| - d_n)$.
From $(|C_n| - d_n)$ to $(|C_n| - d'_n)$ we lose 50%, so the norm $\log(|C| - d_n)/\min J_n!!$ changes by at most $1/\min J_n!!$.

Unhalving = When you lose “all” your money, remember your secret stash and recover it. You are now almost as rich as before. (Logarithmically speaking, at most one digit less.)
Concretely: go back from d'_n to d_n.
Technical lemma: If you apply unhalving to finitely many creatures of a condition q, resulting in a condition q', then $q' =^* q$.

Continuous reading, using halving

We use the lim sup forcing $Q^{\bar{I}}$ which adds a meager set.
("Wlog" we use concrete numbers, for better readability.)

Lemma (Unhalving Lemma)

Let $\underset{\sim}{\alpha}$ be the name of an ordinal.
Given a condition, say $p = (s = \emptyset, (C_0, d_0), (C_1, d_1), \dots)$.
Assume that $C(0)$ allows only 3 possibilities, $C(1)$ allows 10 possibilities, and all norms $\log(|C_n| - d_n)/n!!$ are bigger than 1000 for $n \geq 2$.
Then there is a condition $q \leq p$ such that

- $C_0^q = C_0^p$ *and* $C_1^q = C_1^p$,
- $\forall n \geq 2$: $\log(|C_n^q| - d_n^q)/n!! \geq 970$ *(actually: $\geq 1000 - 30/2!!$)*
- *If there is a condition $r \leq q$, $r = (s_0, s_1, (C_2^r, d_2^r), \dots\dots)$ deciding $\underset{\sim}{\alpha}$, with all norms > 0, then already $q \wedge (s_0, s_1) := (s_0, s_1, (C_2^q, d_2^q), \dots\dots)$ decides $\underset{\sim}{\alpha}$.*

This lemma, rewritten with the proper parameters, allows a fusion argument to show continuous reading for our forcing.

Proof of the unhalving lemma

Start with p. For each possibility s of the 30 possibilities from $C(0) \times C(1)$, say the i-th one, do the following:

- Strengthen the condition by replacing $C(0)$ and $C(1)$ by s.
- ("DECISION") Can you strengthen the current version of $C(2), C(3), \dots$ in such a way that $\underset{\sim}{\alpha}$ is (essentially) decided, but all norms are still $\geq 1000 - i$? If so, do it.
- ("HALVING") Otherwise, apply "halving" to $C(2)$, $C(3)$, etc.

At the end we get a condition q.
Assume that $r = (s_0, s_1, (C_2^r, d_2^r), \dots) \leq q$ decides $\underset{\sim}{\alpha}$. What did we do when we dealt (in step i) with (s_0, s_1)?

- Decided $\underset{\sim}{\alpha}$? Good.
- Halving? Try to get a contradiction.
  Apply unhalving to all those (C_j^r, d_j^r) with norm < 1000 (there are only finitely many) to get a condition $r' =^* r$. But now in r' all creatures have norm $\geq 1000 - i$, so r' witnesses that we were in the DECISION case.

Theorem (Fischer-G-Kellner-Shelah 2015)

Assume GCH, and let κ, λ be regular uncountable.
Let $(I_n : n \in \omega)$ and $(J_n : n \in \omega)$ be as above. (Fast growing sequences of intervals).
Let Q be a product of κ many copies of the "generic null" forcing $Q^{\bar{J}}$ and λ many copies of the "generic meager" forcing $Q^{\bar{I}}$. (not actually true... Use common halving parameter)
Then $\Vdash_Q$ "any set of size $< \kappa$ is null, and any set of size $< \lambda$ is meager".
Moreover: $\Vdash_Q$ non(null)=κ, non(meager)=λ.
Moreover: We can combine this with other forcings (e.g. making $2^{\aleph_0} = \mu$).